Math 144 Discussion Notes
10/13/2009

Problem 1 *The composition of two 1-1 maps is 1-1.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be two 1-1 maps. We want to show that $g \circ f$ is an injective map. Suppose that $g \circ f(a_1) = g \circ f(a_2)$ where $a_1, a_2 \in X$. Then,

$g \circ f(a_1) = g \circ f(a_2)$
$\Longleftrightarrow g(f(a_1)) = g(f(a_2))$
$\overset{g\ 1-1}{\Longrightarrow} f(a_1) = f(a_2)$
$\overset{f\ 1-1}{\Longrightarrow} a_1 = a_2$
$\therefore g \circ f$ is injective. ■

Problem 2 *Suppose that $f : W \to X$ and $g : X \to Y$ are functions such that $g \circ f$ is a bijection. Show that f is 1-1 and g is onto. Suppose that $f : X \to Y$ and $g : Y \to X$ are functions such that $g \circ f$ is the identity map from X onto itself and $f \circ g$ is the identity map from Y onto itself. What can be said about f and g?*

Proof. *Part 1:*
Show f is 1-1:
Suppose $f(a_1) = f(a_2)$.
Then $g \circ f(a_1) = g \circ f(a_2)$.
$\overset{g \circ f \text{ is a bijection}}{\Longrightarrow} a_1 = a_2$
$\therefore f$ is injective.
Show g is onto:
`WTS` $\forall y \in Y\ \exists x \in X$ such that $g(x) = g(x) = y$.
As $g \circ f$ is a bijection $\forall y \in Y\ \exists w \in W$ such that $g(f(w)) = y$.
Given any $y \in Y$, let $x = f(w) \in X$, then $g(x) = y$.
$\therefore g$ is surjective.✓
Part 2:
First note that the identity function is ALWAYS a bijection!
So first since $g \circ f$ is the identity on X, we have that it is a bijection, and thus g is onto and f is 1-1.
Now since $f \circ g$ is the identity on Y, we have that it is also a bijection, and thus f is onto and g is 1-1. So we have that both f and g and both 1-1 and onto, thus they are both bijections. Moreover, they are called *inverses* of each other. ■

Problem 3 *Define a bijection from $8^{\mathbb{N}}$ onto $2^{\mathbb{N}}$.*

Proof. Think of elements in $2^{\mathbb{N}}$ as being written as $a_1, a_2, a_3, \cdots, a_{3n-2}, a_{3n-1}, a_{3n}, \cdots$, where $a_i \in \{0,1\}$.

So we can rewrite any sequence in $2^{\mathbb{N}}$ as $t_1, t_2, \cdots, t_n, \cdots$ where $t_n = (a_{3n-2}, a_{3n-1}, a_{3n})$.

View elements in $8^{\mathbb{N}}$ as being written as $e_1, e_2, e_3, \cdots$, where $e_i \in \{0,1,2,3,4,5,6,7\}$.

Define the map $Bin : \{0,1,2,3,4,5,6,7\} \to \{0,1\}^3$ by $Bin(e_n) = t_n$ where t_n is the binary representation of e_n.

Note:

$$\begin{aligned}
Bin(0) &= (0,0,0)\\
Bin(1) &= (0,0,1)\\
Bin(2) &= (0,1,0)\\
Bin(3) &= (0,1,1)\\
Bin(4) &= (1,0,0)\\
Bin(5) &= (1,0,1)\\
Bin(6) &= (1,1,0)\\
Bin(7) &= (1,1,1)
\end{aligned}$$

One important thing to notice here is that each number in $\{0,1,2,3,4,5,6,7\}$ has a UNIQUE binary representation.

Define the map $BinDecomp : 8^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $e_1, e_2, e_3, \cdots \mapsto t_1, t_2, t_3, \cdots$.

Then, since the binary decomposition of each number is unique, $BinDecomp$ is a bijection. ■

Problem 4 *Prove that $2^{\mathbb{N}}$ is uncountable.*

Proof. Suppose that $2^{\mathbb{N}}$ is countable.

Then we can write its elements as a squence:

$\{a_1, a_2, a_3, \cdots\}$

If it were countable, then all elements of $2^{\mathbb{N}}$ would be listed here.

Each a_i in the sequence above looks like:

$a_i = a_{i1}, a_{i2}, a_{i3}, \cdots$ where $a_{ij} \in \{0,1\}$.

Consider the element $b = b_1, b_2, b_3, \cdots$ where $b_i = \begin{cases} 0 \text{ if } a_{ii} = 1 \\ 1 \text{ if } a_{ii} = 0 \end{cases}$.

Then b is an element of $2^{\mathbb{N}}$ that is not listed in the above sequence.

Thus it is not possible to list every element of $2^{\mathbb{N}}$ in a sequence.

$\therefore 2^{\mathbb{N}}$ is uncountable ■

Problem 5 *Prove that $\mathbb{N}_0^{\mathbb{N}}$ is uncountable.*

Proof. Suppose that $\mathbb{N}_0^{\mathbb{N}}$ is countable.

Then we can write its elements as a squence:

$\{a_1, a_2, a_3, \cdots\}$

If it were countable, then all elements of $\mathbb{N}_0^{\mathbb{N}}$ would be listed here.

Each a_i in the sequence above looks like:

$a_i = a_{i1}, a_{i2}, a_{i3}, \cdots$ where $a_{ij} \in \mathbb{N}_0$.

Consider the element $b = b_1, b_2, b_3, \cdots$ where $b_i = \begin{cases} 0 \text{ if } a_{ii} \neq 0 \\ 1 \text{ if } a_{ii} = 0 \end{cases}$.

Then b is an element of $\mathbb{N}_0^{\mathbb{N}}$ that is not listed in the above sequence.

Thus it is not possible to list every element of $\mathbb{N}_0^{\mathbb{N}}$ in a sequence.

$\therefore \mathbb{N}_0^{\mathbb{N}}$ is uncountable ■

Problem 6 *Show that* $FS\mathbb{N}_0 \approx FS\mathbb{Z}$*.*

Proof. Consider the following ordering of $\mathbb{Z}$:

$\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, -4, 4, \cdots\}$.

Observe the following where $\lambda : \mathbb{N}_0 \to \mathbb{Z}$ and $\eta : \mathbb{Z} \to \mathbb{N}_0$:

$$\begin{array}{ccccccccccccc} \mathbb{N}_0 & = & \{ & 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8 & \cdots & \} \\ (\lambda = \downarrow \quad \eta = \uparrow) & & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots & \\ \mathbb{Z} & = & \{ & 0, & -1, & 1, & -2, & 2 & -3, & 3, & -4, & 4 & \cdots & \} \end{array}$$

This show that both λ and η are bijections and are inverses of each other.

Given any sequence in $FS\mathbb{N}_0$ we can use the map λ to uniquely change the entries from being in $\mathbb{N}_0$ to being in $\mathbb{Z}$ and thus attaining a sequence in $FS\mathbb{Z}$, and vice-versa using η.

Thus the map given by $\Lambda : FS\mathbb{N}_0 \to FS\mathbb{Z}, n_1, n_2, n_3, \cdots \mapsto \lambda(n_1), \lambda(n_2), \lambda(n_3), \cdots$ is a bijection (its inverse is given by $\Lambda^{-1} : FS\mathbb{Z} \to FS\mathbb{N}_0,\ z_1, z_2, z_3, \cdots \mapsto \eta(n_1), \eta(n_2), \eta(n_3), \cdots$).

$\therefore FS\mathbb{N}_0 \approx FS\mathbb{Z}$ ■

Problem 7 *Show that* $\mathbb{N} \approx \mathbb{Q}^+$*.*

Proof. We want a bijection between $\mathbb{N}$ and $\mathbb{Q}^+$.

We have the bijections:

$$\begin{aligned} Decomp_{\mathbb{N}_0} &: \mathbb{N} \to FS\mathbb{N}_0 \\ \Lambda &: FS\mathbb{N}_0 \to FS\mathbb{Z} \\ Comp_{\mathbb{Q}^+} &: FS\mathbb{Z} \to \mathbb{Q}^+. \end{aligned}$$

Since we know that the composition of bijections is a bijection we have:

$$Comp_{\mathbb{Q}^+} \circ \Lambda \circ Decomp_{\mathbb{N}_0} : \mathbb{N} \to \mathbb{Q}^+$$

which is a bijection.

$\therefore \mathbb{N} \approx \mathbb{Q}^+$ ■